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Classical Solution of Post-buckled nonlinear Beam on an Elastic Foundation

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ABSTRACT

This paper is concerned with the buckling problem of flexible beams on an elastic foundation for free vibration. An exact solution for the post-buckled geometric nonlinear beam with clamped-clamped and clamped-hinged end conditions is presented in this paper. The cubic nonlinearity of the governing equation of motion is induced due to the mid-plane stretching, which is considered in the analysis. The critical buckling forces are obtained as 4.015, 4.061 and 4.122 with respect to nondimensional foundation stiffness 1,10 and 20, respectively, with the fundamental mode of vibration. The critical buckling load, associated mode shape, the effect of foundation stiffness, and vibration behavior are obtained. The optimum locations of an internal hinge and the optimum buckling force are also investigated for various foundation stiffness of the nonlinear beam on an elastic foundation. The bifurcation diagrams and the internal hinge locations are useful for practical application of axially loaded nonlinear beam on an elastic foundation restrained by Clamped-Clamped and Clamped-Hinge beam.

1. Introduction

Buckling is a static instability of structures due to in-plane loading and solving the nonlinear buckling problem for a given axial load results in the post buckling configurations [1]. A plenty of research has been carried out on the subject of buckled beam [2-8], plates [9-10] and rods [11-13] for many years. Among them, Nayfeh et al. (1995) [3] and Chen (1994) [9] were formulated the static buckled configurations to obtain buckled shapes and their associated natural frequencies with the fixed and simply supported post-buckled beams. The governing equations of the nonlinear buckled beams were induced a geometric nonlinearity in most of the research. The geometric nonlinearity is due to the mid-plane stretching, which is taken into account in the present study. Fang and Wickert (1994) [2] studied the static deformation of micro-machined beams under prescribed in-plane compressive stress by analytical and experimental means based on geometrically nonlinear imperfect beam. Addessi et al. (2005) [4] investigated free un-damped in-plane vibrations of shear un-deformable beams around their highly buckled configuration neglecting rotary

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inertia effects. Zhang et al. (2005) [5] investigated the secondary bifurcations and tertiary states of a beam resting on nonlinear foundation. They used a three mode Galerkin discretization to produce a set of nonlinear algebraic equilibrium equations and then the algebraic equations were solved by numerically using the root solving and pseudo-arch-length method.

In the recent years, Li and Batra (2007) [6] studied the ling and the post buckling deformation of uniformly heated pinned-pinned and fixed-fixed Euler-Bernoulli beams supported on linear elastic foundations. They used the shooting method to compute the buckling modes and transitions among them by solving analytically the linear problem. Recently, Xia et al. (2010) [8] investigated the static bending, post buckling and free vibration of nonlinear micro-beams. This study established a nonlinear non-classical Euler-Bernoulli beam model for micro-scale beam by using the modified couple stress theory. Jiao et al. (2017) [14] developed a large deformation model for static and dynamic analysis of the post buckled beams to investigate the buckling response of the slender beams. Then, they used the energy balance method to solve the nonlinear governing equations. Rammerstorfer (2018) [15] presented the new results of bifurcation buckling under tensile loading of beams and plates. Later on, Zare and Asnafi (2019) [16] had caried out the analysis of nonlinear pre and post buckled curved beams. They used the differential quadrature element method for solving of nonlinear equations. Neukirch et al. (2021) [17 were made a comparison of the Von Karman and Kirchhoff models between the post buckling and vibration of elastic beams.

Buckling of column/beam is basic in elastic stability. In some cases, beam may have to provide interior joints or internal hinge. The internal hinge may be necessary in designs to facilitate the opening of doors and hatches or other swivel motions. Previously, the buckling force and optimum hinge location on fundamental frequency had been investigated on beams [18-20] and plates [21-23]. Also, exact vibration solutions of structural members were summarized by Wang et al. (2005) [24]. In the recent years, buckling of column [25] and an infinite beam [26] with the internal hinge attached to an elastic foundation have been investigated. Most the works available in the literature for determination of internal hinge location of a beam is the linear vibration problem except Cheng et al. (2003) [20] is a nonlinear random response. Nonlinear vibration of buckling problems of beam with an internal hinge is rare in the literature. The aim of the present study is to determine the optimum location of internal hinge and critical buckling forces at various foundation stiffness of nonlinear beam for clamped-clamped (C-C) and clamped-hinged (C-H).

In this study, an exact solution of the governing differential equation is presented. The geometric nonlinearity is governed due to the mid-plane stretching of the beam; as a result, the governing equation is formulated with a cubic nonlinearity. Two types of end conditions of the geometric nonlinear beam such as C-C and C-H on an elastic foundation are taken into account for the analysis of the buckling problem. Exact vibration solutions for internal hinge locations and optimum buckling force corresponding to various foundations stiffness are also investigated for C-C and C-H nonlinear post-buckled beam on an elastic foundation.

2. Theory

The governing equation of motion of nonlinear vibration of the Euler-Bernoulli beam on elastic foundation including the effect of mid-plane stretching [27] is as follows in Eq. (1).

$$EI\widehat{W}'''' + m\ddot{\widehat{W}} + \hat{\xi}\dot{\widehat{W}} + \hat{F}\widehat{W}'' + \widehat{K}_f\widehat{W} - \left(\frac{EA}{2L}\int_0^L (W')^2 d\hat{x}\right)\widehat{W}'' = \hat{P}\cos(\hat{\Omega}\hat{t})$$

Where, the prime indicates the derivative with respect to $\hat{x}$, over dot indicates the derivative with respect to $\hat{t}$ and $\widehat{W}$ denotes the transverse displacement by the mid-plane stretching of the beam on elastic foundation. The m is the mass per unit unreformed length, cross section area A, moment of inertia I, length of the beam L, Young's modulus of the beam E, damping coefficient of the beam ξ, foundation coefficient of modulus K_f, axial force acting on the beam $\hat{F}$, excitation amplitude $\hat{P}$, excitation frequency $\widehat{\Omega}$. For the convenience, the following non-dimensional variables are used as presented in Eq. (2).

$$x = \frac{\hat{x}}{L}, W = \frac{\widehat{W}}{r}, t = \hat{t}\sqrt{\frac{EI}{mL^4}}, F = \frac{\hat{F}L^2}{EI}, \xi = \frac{\hat{\xi}L^2}{\sqrt{mEI}}, r = \sqrt{\frac{I}{A}}, K_f = \frac{\widehat{K}_f L^4}{EI}, \text{ and } P = \frac{\hat{P}L^4}{rEI}$$

Where, *r* is the radius of gyration of the cross section of the beam, therefore, Eq. (1) can be written in Eq. (3).

$$W'''' + \ddot{W} + \xi \dot{W} + FW'' + K_f W - \frac{1}{2} W'' \int_0^L (W')^2 \, dx = Psin(\Omega t)$$

The associated boundary conditions for *C-C* and *C-H* beam are as written in Eq. (4) and Eqs. (5a-5b).

$W = 0$ and $W' = 0$ at $x = 0, L$
$W = 0$ and $W' = 0$ at $x = 0$
$W = 0$ and $W'' = 0$ at $x = L$

2.1. Buckling formulation

Consider the time dependent, damping factor and force terms are zero, the buckling problem can be obtained from Eq. (3) is as follow in Eq. (6).

$$W''''(x) + FW''(x) + K_f W(x) - \frac{1}{2} W''(x) \int_0^L (W'(x))^2 \, dx = 0 \tag{6}$$

2.2. Exact Solution

The integral is constant [1] in Eq. (6) for given $W(x)$. So, consider Q denotes this constant and as presented in Eq. (7).

$$Q = \frac{1}{2} \int_0^L (W')^2 \, dx$$

Substituting Eq. (7) into Eq. (6), the results can be expressed as shown in Eq. (8).

$$W'''' + \lambda W'' + K_f W = 0$$

Where $\lambda = F - Q$ represents the critical buckling load and Eq. (8) is a fourth order ordinary differential equation whose general solution can be expressed into three types [25].

Case 1, if $\lambda^2 > 4K_f$, the solution can be written as in Eqs. (9a-9b).

$$W(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x) + C_3 \sin(\beta x) + C_4 \cos(\beta x)$$

Where,

$$\alpha = \sqrt{\frac{\lambda - \sqrt{\lambda^2 - 4K_f}}{2}} \text{ and } \beta = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 4K_f}}{2}} \tag{9b}$$

Case 2, if $\lambda^2 = 4K_f$, the solution can be written as in Eqs. (9c-9d).

$$W(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x) + xC_3 \sin(\alpha x) + xC_4 \cos(\alpha x)$$

Where,

$$\alpha = \sqrt{\frac{\lambda}{2}} \tag{9d}$$

Case 3, if $\lambda^2 < 4K_f$, the solution can be written as in Eqs. (9e-9f).

$$W(x) = C_1 e^{-\alpha x} \sin(\beta x) + C_2 \, e^{-\alpha x} \cos(\beta x) + C_3 \, e^{\alpha x} \sin(\beta x) + C_4 \, e^{\alpha x} \cos(\beta x)$$

Where,

$$\alpha = (K_f)^{\frac{1}{4}} \cos\left(\frac{\theta}{2}\right), \ \beta = (K_f)^{\frac{1}{4}} \sin\left(\frac{\theta}{2}\right), \text{ and } \theta = \pi - \tan^{-1}\left(\frac{\sqrt{\lambda^2 - 4K_f}}{\lambda}\right) \tag{9f}$$

In this study we consider only the case 1, $\lambda^2 > 4K_f$, therefore the general solution of the different types of end conditions of the nonlinear beam are as follows:

2.3. *C-C* beam

Applying the boundary conditions of Eq. (4) for clamped-clamped beam, we have Eqs. (10-13).

$$C_2 + C_4 = 0$$
$$\alpha C_1 + \beta C_3 = 0$$
$$C_1 \sin(\alpha) + C_2 \cos(\alpha) + C_3 \sin(\beta) + C_4 \cos(\beta) = 0$$
$$\alpha C_1 \cos(\alpha) - \alpha C_2 \sin(\alpha) + \beta C_3 \cos(\beta) - \beta C_4 \sin(\beta) = 0$$

The determinant of the coefficient matrix of Eqs. (10) – (13) represent the characteristic equations. Therefore, the following characteristic equation can be obtained as in Eq. (14).

$$2\alpha\beta - 2\alpha\beta \cos(\alpha)\cos(\beta) - (\alpha^2 + \beta^2)\sin(\alpha)\sin(\beta) = 0$$

The Eigenvalues are determined by solving the Eq. (14). Now the mode shapes are given by Eq. (15).

$$W(x) = d\left[-\left(\frac{\beta}{\alpha}\right)\sin(\alpha x) - \frac{\alpha \sin(\beta) - \beta \sin(\alpha)}{\alpha(\cos(\alpha) - \cos(\beta))}\cos(\alpha x) + \sin(\beta x) + \frac{\alpha \sin(\beta) - \beta \sin(\alpha)}{\alpha(\cos(\alpha) - \cos(\beta))}\cos(\beta x)\right]$$

Where, d is a constant to be determined. The expression of mode shapes $W(x)$ governs both the symmetric and anti-symmetric mode shapes. The buckle configuration $W(x)$ satisfies the boundary condition as well as the following condition due the mid stretching of the beam and presented in Eq. (16).

$$\lambda = F - Q = F - \frac{1}{2}\int_0^L (W')^2 dx \tag{16}$$

Substituting Eq. (15) into Eq. (16), a relationship with the variables F, K_f, d and λ are obtained. Therefore, for a given axial load F, the constant d corresponding to any λ can be determined, and subsequently, the mode shapes of beam can be obtained by using the Eq. (15). Details are shown in Appendix.
Simplify the Eq. (14) the symmetric mode is as follows that is expressed in Eqs. (17a-17b).

$$(\alpha + \beta)\sin\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) + (\alpha - \beta)\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = 0 \tag{17a}$$
$$\tan\left(\frac{\alpha}{2}\right) = \left(\frac{\beta}{\alpha}\right)\tan\left(\frac{\beta}{2}\right) \tag{17b}$$

Simplify the Eq. (14) the anti-symmetric mode is as follows that is expressed in Eqs. (18a-18b).

$$(-\alpha - \beta)\sin\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) + (\alpha - \beta)\sin\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = 0 \tag{18a}$$
$$\tan\left(\frac{\alpha}{2}\right) = \left(\frac{\alpha}{\beta}\right)\tan\left(\frac{\beta}{2}\right) \tag{18b}$$

2.4. *C-H* beam

Similarly, satisfying the boundary conditions Eq. (5) the characteristic equation for the clamped-hinged beam can be written as in Eq. (19).

$$(\alpha^2 - \beta^2)(\alpha\cos(\alpha)\sin(\beta) - \beta\cos(\beta)\sin(\alpha)) = 0 \tag{19}$$

The Eigenvalues are determined by solving the Eq. (19). Again, the mode shapes yield Eq. (15) and the critical buckling load can be determined after determining the constant d by using the Eq. (19). The characteristic equations and Eigenvalues for various end conditions of beam on elastic foundation are summarized in Table 1.

Table 1: The Eigen values for various end conditions of beam on elastic foundation

End conditions of beam	Eigen values when $\beta = 0.3$ by using Eqs. (14) and (19) for *C-C* and *C-H* beams, respectively
C-C	6.298, 8.971, 12.575, 15.451, 18.858
C-H	4.483, 7.752, 10.889, 14.084, 17.212

2.5. Internal hinge location of beam

Consider the internal hinge is at $x = a$ and the subscripts 1, 2 represent the sections $0 \ll x \ll a$ and $a \ll x \ll 1$, respectively of the beam. The conditions for internal hinge location are that the deflections are continuous, the moments are zero and the shear forces are equal [18; 25]. Therefore, the expressions are presented in Eqs. (20-23).

$$W_1(x) = W_2(x) \tag{20}$$
$$W_1''(x) = 0 \tag{21}$$
$$W_2''(x) = 0 \tag{22}$$
$$W_1'''(x) + \lambda W_1(x) = W_2'''(x) + \lambda W_2(x) \tag{23}$$

Consider both ends of the beam are clamped, the solutions $W(x)$ satisfying the boundary condition $W_1(x) = W_1'(x) = 0$ are expressed in Eqs. (24-25).

$$W_1(x) = C_1[\beta\sin(\alpha x) - \alpha\sin(\beta x)] + C_2[\cos(\alpha x) - \cos(\beta x)] \tag{24}$$
$$W_2(x) = C_3[\beta\sin(\alpha(x-1)) - \alpha\sin(\beta(x-1))] + C_4[\cos(\alpha(x-1)) - \cos(\beta(x-1))] \tag{25}$$

Satisfying the four conditions in Eqs. (20-23), Eq. (24) and Eq. (25) can be written as follows in Eqs. (26-29).

$$C_1 = -C_2\left[\frac{\alpha^2\cos(\alpha x) - \beta^2\cos(\beta x)}{\alpha\beta(\alpha\sin(\alpha x) - \beta\sin(\beta x))}\right] \tag{26}$$

$$C_3 = -C_4\left[\frac{\alpha^2\cos(\alpha(x-1)) - \beta^2\cos(\beta(x-1))}{\alpha\beta(\alpha\sin(\alpha(x-1)) - \beta\sin(\beta(x-1)))}\right] \tag{27}$$

$$C_1[\beta\sin(\alpha x) - \alpha\sin(\beta x)] + C_2[\cos(\alpha x) - \cos(\beta x)] = C_3[\beta\sin(\alpha(x-1)) - \alpha\sin(\beta(x-1))] + C_4[\cos(\alpha(x-1)) - \cos(\beta(x-1))] \tag{28}$$

$$C_1(-\alpha^3\beta\cos(x\alpha) + \alpha\beta^3\cos(x\beta)) + C_2(\alpha^3\sin[x\alpha] - \beta^3\sin(x\beta)) + \lambda(C_1(\alpha\beta\cos(x\alpha) - \alpha\beta\cos(x\beta)) + C_2(-\alpha\sin(x\alpha) + \beta\sin(x\beta))) = C_3(-\alpha^3\beta\cos[(-1+x)\alpha] + \alpha\beta^3\cos[(-1+x)\beta]) + C_4(\alpha^3\sin[(-1+x)\alpha] - \beta^3\sin[(-1+x)\beta]) + \lambda(C_3(\alpha\beta\cos[(-1+x)\alpha] - \alpha\beta\cos[(-1+x)\beta]) + C_4(-\alpha\sin[(-1+x)\alpha] + \beta\sin[(-1+x)\beta])) \tag{29}$$

Eqs. (31-34) are used to set up the Eigen-value problem which will take the form of $\mathrm{KC} = 0$.
Where, $C^T = \{C_1, C_2, C_3, C_4\}$. The vanishing of the determinant of the matrix K yields the buckling force. Hence, the lowest Eigenvalue λ is the buckling load. It can be obtained by the bisection method for given values of a, K_f and F.

Consider one end clamped and the other end hinge, the solutions $W(x)$ satisfying the conditions of the Eqs. (20-23) we have Eqs. (30-34).

$$W_1(x) = C_1(\beta\sin[x\alpha] - \alpha\sin[x\beta]) + C_2(\cos[x\alpha] - \cos[x\beta]) \tag{30}$$

$$W_2(x) = C_3(\sin[\alpha(-1+x)]) + C_4(\sin[\beta(-1+x)]) \tag{31}$$

$$C_1 = -C_2\left[\frac{\alpha^2\cos(\alpha x) - \beta^2\cos(\beta x)}{\alpha\beta(\alpha\sin(\alpha x) - \beta\sin(\beta x))}\right] \tag{32}$$

$$C_3 = -C_4\left[\frac{\beta^2\sin(\beta(x-1))}{\alpha^2\sin(\alpha(x-1))}\right] \tag{33}$$

$$\begin{aligned} &C_1(-\alpha^3\beta\text{Cos}[x\alpha] + \alpha\beta^3\text{Cos}[x\beta]) + C_2(\alpha^3\text{Sin}[x\alpha] - \beta^3\text{Sin}[x\beta]) \\ &\qquad + \lambda\big(C_1(\alpha\beta\text{Cos}[x\alpha] - \alpha\beta\text{Cos}[x\beta]) + C_2(-\alpha\text{Sin}[x\alpha] + \beta\text{Sin}[x\beta])\big) = \\ &-C_3\alpha^3\text{Cos}[(-1+x)\alpha] - C_4\beta^3\text{Cos}[(-1+x)\beta] + \lambda(C_3\alpha\text{Cos}[(-1+x)\alpha] + C_4\beta\text{Cos}[(-1+x)\beta] \end{aligned} \tag{34}$$

3. Results and discussion

The nonlinear post-buckled vibrations of beam on an elastic foundation are analyzed with *C-C* and *C-H* (see Figure 1). The results are presented in the following sections with non-dimensional parameters such as length of the beam, axial force, static deflection, and foundation stiffness.

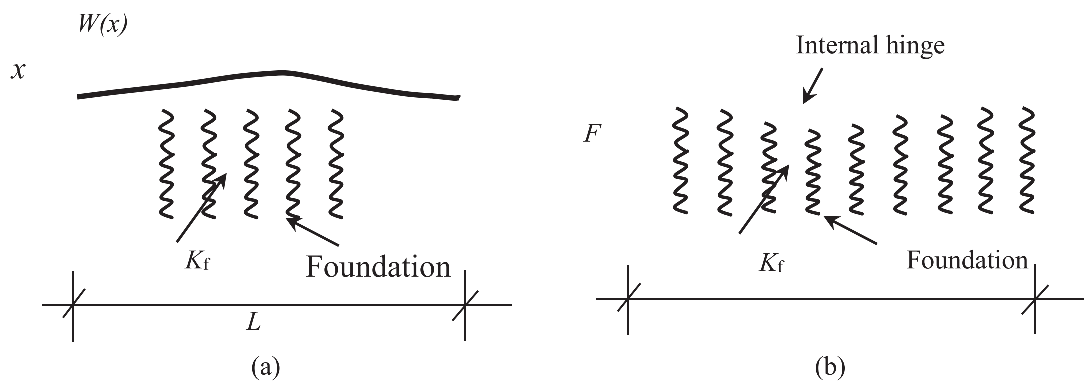
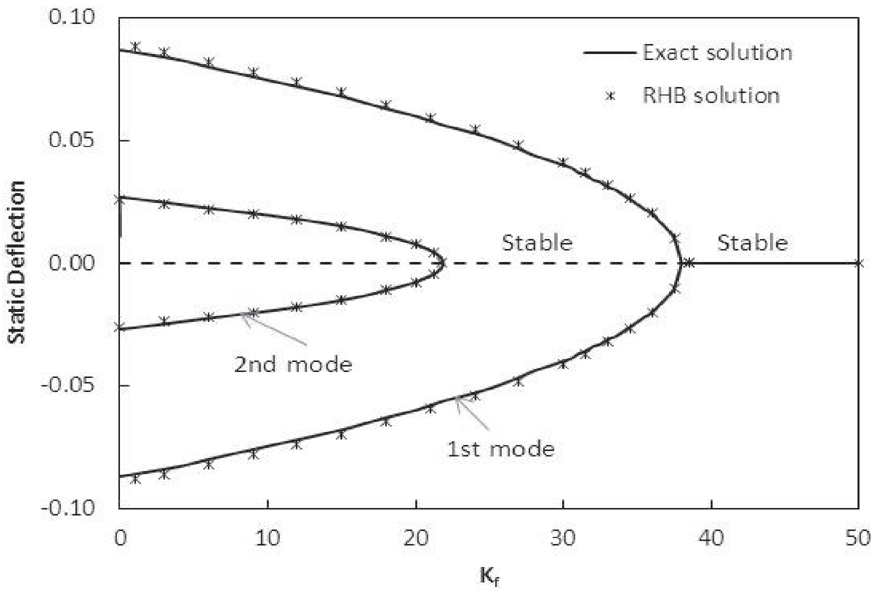
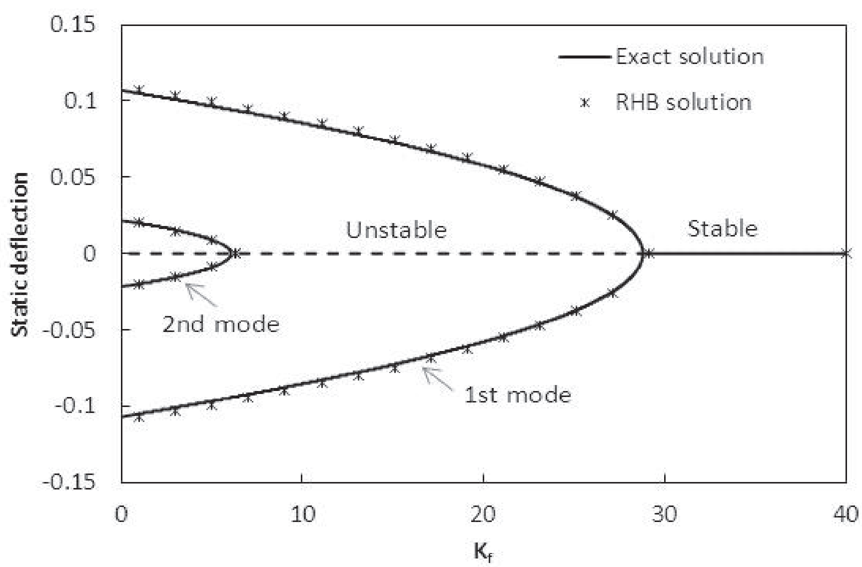


Figure 1: A flexible beam on elastic foundation subjected to the axial load, (a) without a and (b) with internal hinge.

Figure 2 illustrates the non-dimensional static deflection with respect to various foundations for C-C and C-H beam. The first two modes bifurcation diagrams for non-dimensional static deflection as a function of non-dimensional foundation stiffness are presented in Figures 2(a-b). The non-dimensional static deflections are plotted of the point at $x = 0.3L$. As the foundation stiffness decreases from the first critical stiffness at K_f = 38.00 and 28.76, the straight position loses stability and therefore, the buckling is formulated for C-C and C-H beam respectively in Figures 2(a-b). It can be seen that Figure 2(b) is less capacity to carry the critical force than the Figure 2(a) with respect to stiffness K_f.The obtained results compared with residue harmonic balance method (RHB) which shows good agreement. The stability analysis is not included in this study.



(a)

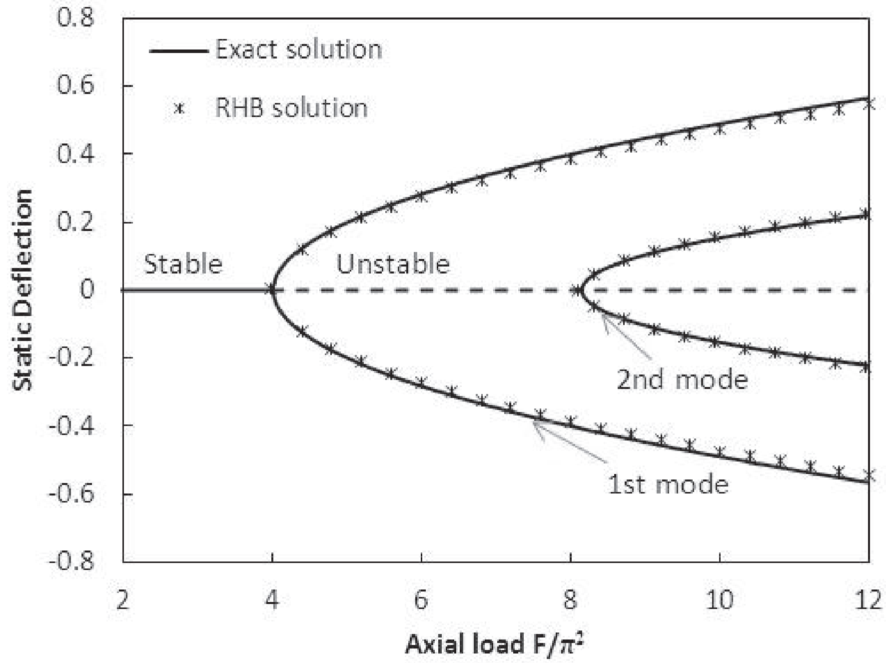


(b)

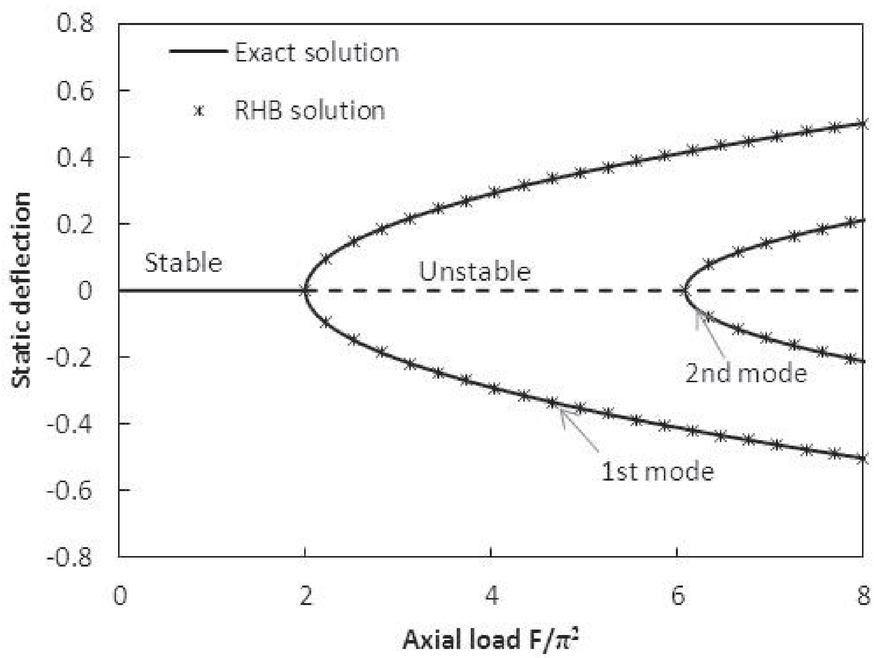
Figure 2: Non-dimensional static deflection corresponding to foundation stiffness of 1st and 2nd mode (a) *C-C* beam and (b) *C-H* beam.

Figure 3 shows the non-dimensional critical buckling load obtained from exact solution for *C-C* and *C-H* beam. The first two modes bifurcation diagrams for non-dimensional deflection as a function of non-dimensional axial force are presented in Figures 3(a-b). The non-dimensional static deflections are plotted at $x = 0.3L$ with the foundation stiffness when $K_f = 1$. As the axial load increase from the 1st mode critical buckling load at $F/\pi^2 = 4.105$ and 2.038, the straight position loses stability and thus, the buckling is formulated for *C-C* and *C-H* beam in Figures 3(a-b). The obtained results well agree with results obtained by RHB method. The stability analysis is not included in this study.

Figure 4 shows bifurcation diagrams of the non-dimensional critical buckling loads obtained from exact solution method for various foundation stiffness at 1st mode vibration of *C-C* and *C-H* beams. The results of axial critical force obtained for various foundations stiffness from 1st to 3rd mode is presented in Tables 2(a-b). It can be seen that the critical axial force is increasing with increasing the foundation stiffness in Tables 2(a-b) for *C-C* and *C-H* beam, respectively. The obtained results show good agreement with results obtained by RHB method.

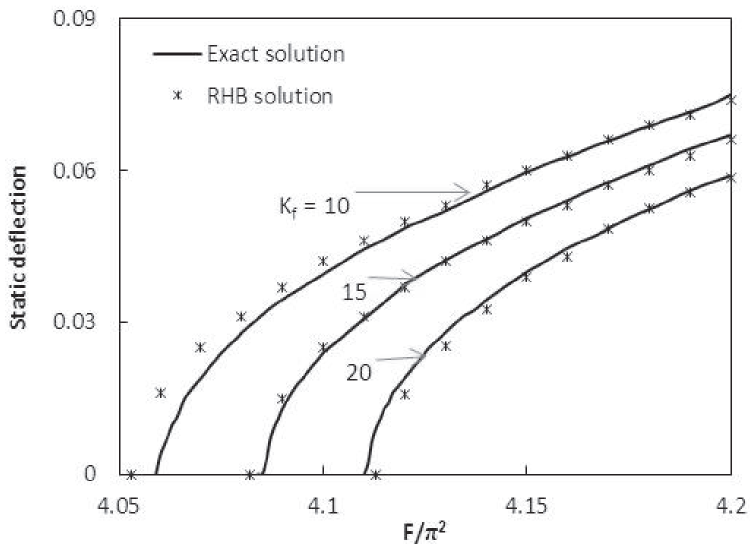


(a)

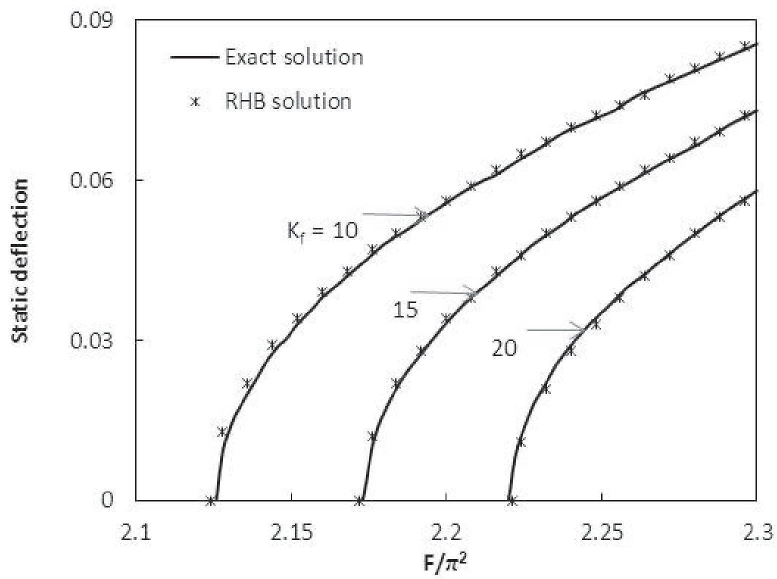


(b)

Figure 3: Non-dimensional static deflection corresponding to axial load of 1st and 2nd mode (a) *C-C* beam and (b) *C-H* beam.



(a)



(b)

Figure 4: Non-dimensional static deflection corresponding to foundation stiffness at 1st mode vibration (a) *C-C* beam and (b) *C-H* beam.

Table 2(a): The critical buckling force at various K_f for *C-C* beam

K_f	Axial load $\frac{F}{\pi^2}$ at $x = 0.3L$		
	1st mode	2nd mode	3rd mode
1	4.015	8.148	16.014
5	4.035	8.158	16.019
10	4.061	8.170	16.026
15	4.086	8.183	16.032
20	4.112	8.195	16.039

Figure 5 shows the results of non-dimensional static deflection corresponding to the velocity of 2nd mode vibration for *C-C* and *C-H* beams when K_f = 1. The phase diagram is plotted of the point at $x = 0.3L$. The phase diagram of 2nd mode vibration attributes more like a kidney shape with a closed trajectory for *C-C* beam in Figure 5(a). The kidney shape of the curves demonstrates that the nonlinearity dominates the vibration behavior of *C-C* post-buckled beam on elastic foundation at the 2nd mode vibration. The period doubling phenomenon can be seen in case of *C-H* beam at 2nd mode vibration along with a non-closed end in Figure 5(b). The closed and non-closed trajectories are mainly for the boundary condition of the *C-C* and *C-H* beam. The nature of the curves notices that loading effect is more dominant than the nonlinearity for *C-H* post-buckled beam.

Table 2(b): The critical buckling force at various K_f for *C-H* beam

K_f	Axial load $\frac{F}{\pi^2}$ at $x = 0.3L$		
	1st mode	2nd mode	3rd mode
1	2.038	6.083	12.006
5	2.078	6.097	12.013
10	2.127	6.113	12.022
15	2.174	6.130	12.030
20	2.221	6.147	12.039

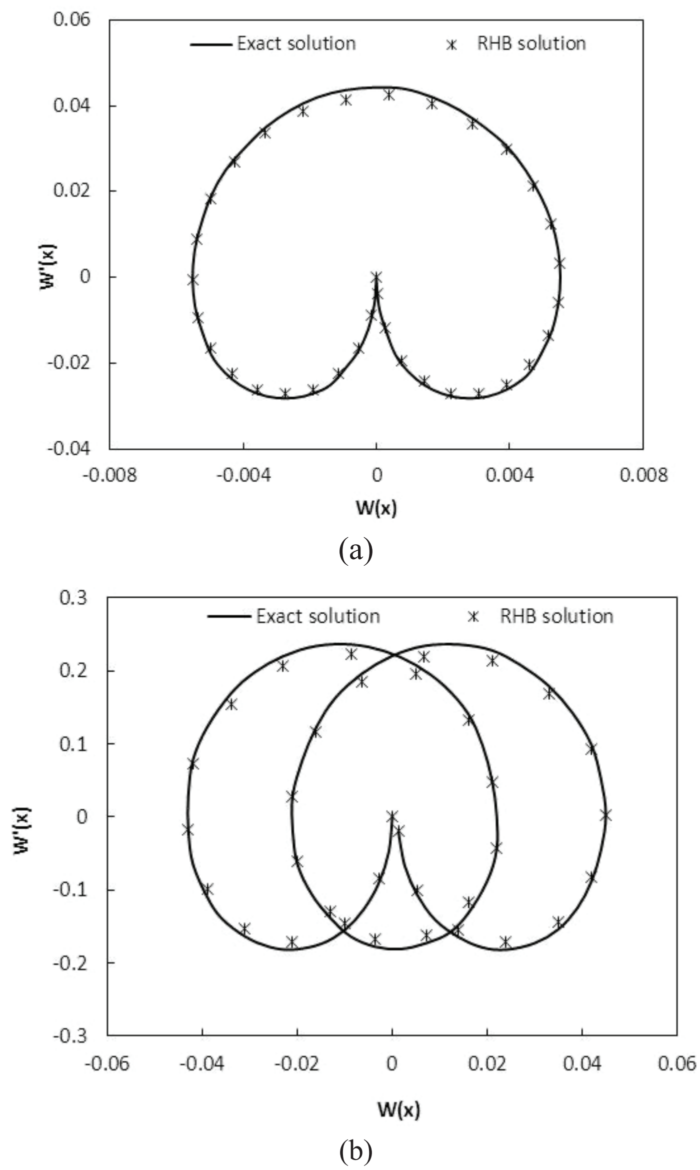


Figure 5: Phase diagram of post buckle beam at 1st mode vibration when $K_f = 1$, (a) *C-C* beam, and (b) *C-H* beam.

Figure 6 illustrates that the phase diagram corresponding to different foundation stiffness of post-buckled beam at 1st mode vibration for *C-C* and *C-H* beams. The phase diagrams are again plotted of the point at $x = 0.3L$. The phase diagram of the *C-C* beam has a closed trajectory, and the shape of the curves look like an ellipse at the 1st mode vibration in Figure 6(a). As the value of the foundation stiffness is increased, the trajectory of motion of nonlinear *C-C* beam system is reduced a small ellipse. It is clear that the displacement of the system gradually decreases from K_f = 1 to K_f = 15 and rapidly decease at K_f = 20 in Figure 6(a). The phase diagram of the nonlinear *C-C* beam in all cases of the foundation stiffness are horizontally i.e., displacement or static deflection axis symmetric closed curve with a fixed origin and their centre are in the different points.

On the other hand, the phase diagram of *C-H* beam (Figure 6(b)) has a non-closed trajectory, and the shape of the curve is more likely kidney shape. The nature of the phase diagram of the *C-H* beam demonstrates that the nonlinearity dominates in the system of *C-H* beam on elastic foundation at 1st mode vibration. Moreover, the trajectory of the *C-H* beam originates at the centre of the displacement and velocity; and the trajectory is vertically i.e., velocity axis symmetric. In addition, there is a common or fixed centre of the phase trajectories in the all cases of foundation stiffness for *C-H* beam.

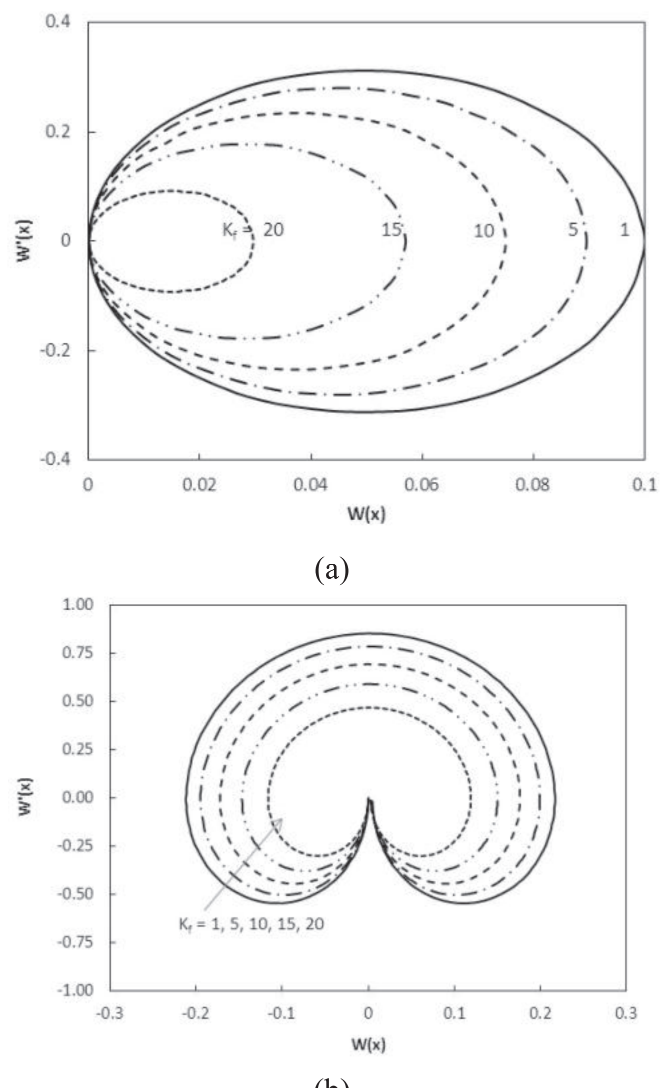


Figure 6: Phase diagram of post buckle beam at 2nd mode vibration when K_f = 1, (a) *C-C* beam, and (b) *C-H* beam.

Figure 7 presents the internal hinge location of a nonlinear *C-C* beam with optimum non-dimensional axial force regarding to various foundation stiffness. The results of the *C-C* beam are symmetric about the mid-point at $a = 0.5$, hence the results for hinge location up to $a = 0.5$ are presented in this figure. Note that the mid-plane stretching loads Q are determined by using the Eqs. (A2.1-A2.3) corresponding to obtained critical axial loads (i.e., from bifurcation analysis) for the specified foundation stiffness. Then, the Figure 7 is plotted by using Eqs. (26-29). It can be seen that the optimum non-dimensional buckling forces increase with the increase of foundation stiffness. For the lower value of foundation stiffness, the optimum location of internal hinge is about near the clamped supported end of both side of the beam. As the foundation stiffness increase with an equal interval, the optimum locations of internal hinge are shifted with an equal distance of interval from clamped support to inside of the point a, please see more details in Table 3(a). The obtained results of internal hinge location with the optimum non-dimensional buckling force for *C-C* beam are presented in Table 2(a) corresponding to various foundation stiffness (one side only).

Figure 8 illustrates the internal hinge location of a nonlinear *C-H* beam with optimum non-dimensional axial force regarding to various foundation stiffness. Note that the mid-plane stretching loads Q are determined again by using the Eqs. (A2.1-A2.3) corresponding to obtained critical axial loads (i.e., from bifurcation analysis) for the specified foundation stiffness. Then, the Figure 8 is plotted by using Eqs. (30-34). Figure 8 shows that the optimum locations of the hinge are shifted from the clamped support to the inside of the beam when K_f = 10 to 20. As the foundation stiffness increases from K_f = 20 to 30, the optimum locations of the hinge are shifted from the inside to the clamped support of the beam. The obtained results of internal hinge location with the optimum non-dimensional buckling force for *C-H* beam are presented in Table 3(b) corresponding to various foundation stiffness.

Table 3(a): The internal hinge location with optimum non-dimensional buckling force for *C-C* beam

K_f	20	30	35	40	45
a_{opt}	0.061	0.081	0.091	0.101	0.111
λ_{max}	19.972	31.027	36.100	40.598	44.347

Table 3(b): The internal hinge location with optimum non-dimensional buckling force for *C-H* beam

K_f	10	15	20	25	30
a_{opt}	0.799	0.781	0.768	0.772	0.792
λ_{max}	8.818	13.724	17.643	20.568	23.249

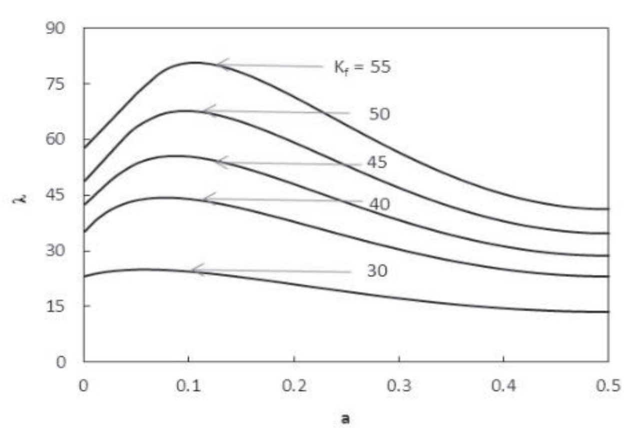


Figure 7: The internal hinge location on various foundations stiffness of *C-C* nonlinear beam with optimum non-dimensional buckling force.

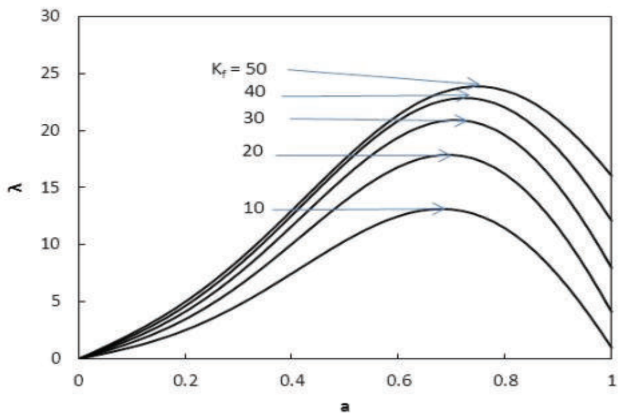


Figure 8: The internal hinge location corresponding to various foundation stiffness of *C-H* beam nonlinear with optimum non-dimensional buckling force.

4. Conclusions

Classical solution is presented to solve the nonlinear vibrations of post-buckled beam on an elastic foundation with *C-C* and *C-H* end conditions. The effect of foundation stiffness, critical buckling force and interesting vibration behaviors are investigated. The exact vibration solutions for axially loaded nonlinear beams on an elastic foundation with an internal hinge are obtained. The optimum non-dimensional buckling forces are investigated corresponding to the different foundation stiffness for *C-C* and *C-H* beam. The result shows that the foundation stiffness is greatly influenced to the buckling force of the beam with an internal hinge. The result obtained from the bifurcation diagrams and the internal hinge locations are useful for practical application with such kind of axially loaded nonlinear beam on an elastic foundation.

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References

[1] Nayfeh A.H. Emam S.A., “Exact solution and stability of post buckling configurations of beams”, *Nonlinear Dyn,* vol. 54, pp. 395-408, 2008.

[2] Fang W. Wickert J.A., “Post buckling of micromachined beams”, *J. Miromech. Microeng,* vol. 4, pp. 116-122, 1994.

[3] Nayfeh A.H. Kreider W. Anderson T.J., “Investigation of natural frequency and mode shapes of buckled beam”, *AIAA Journal*, vol. 33, no. 6, pp. 1121-1126, 1995.

[4] Addessi A. Lacarbonara W. Paolone A., “Free in-plane vibrations of highly buckled beams carrying a lumped mass”, *Acta Mechanica*, vol. 180, pp. 133-156, 2005.

[5] Zhang Y. Murphy K.D., “Secondary buckling and tertiary states of a beam on a non-linear elastic foundation”, *International Journal of Non-linear Mechanics,* vol. 40, pp.795-805, 2005.

[6] Li S.R. Batra R.C., “Thermal buckling and post buckling of Euler-Bernoulli beams supported on nonlinear elastic foundations”, *AIAA Journal*, vol. 45, No. 3, pp. 712-720, 2007.

[7] Emam S.A. Nayfeh A.H., “Post buckling and free vibrations of composite beams”, *Composite Structures*, vol. 88, pp.636-642, 2009.

[8] Xia W. Wang L. Yin L., “Nonlinear non-classical microscale beams: Static bending, post buckling and free vibration”, *International Journal of Engineering Science*, vol. 48, pp. 2044-2053, 2010.

[9] Chen H.P., “Free vibration of parbuckled and post buckled plates with delamination”, *Composites Science and Technology,* vol. 51, pp. 451-462, 1994.

[10] Librescu L. Lin W., “Post buckling and vibration of shear deformable flat and curved panels on a non-linear elastic foundation”, *International Journal of Non-linear Mechanics*, vol. 32, no. 2, pp. 211-225, 1997.

[11] Whiting A.I.M., “A Galerkin procedure for localized buckling of a strut on a nonlinear elastic foundation”, *International Journal of Solids and Structures*, vol. 34, no. 6, pp. 727-739, 1997.

[12] Wang C.Y., “Post-buckling of a clamped-simply supported elastica”, *International Journal of Non-linear Mechanics*, vol. 32, no. 6, pp. 1115-1122, 1997.

[13] Li S. Zhou Y.H. Zheng X., “Thermal post buckling of a heated elastic rod with pinned-fixed ends”, *Journal of Thermal Stresses*, vol. 25, pp. 43-56, 2002.

[14] Jiao P. Borchani W. Lajnef N., “Large deformation solutions to post-buckled beams confined by movable and flexible constraints: A static and dynamic analysis”, *International Journal of Solids and Structures,* vol. 128, pp. 85-98, 2017.

[15] Rammerstorfer F.G., “Buckling of elastic structures under tensile loads”, *Acta Mech*, vol. 229, pp. 881-900, 2018.

[16] Zare M. Asnaf A., “Nonlinear pre- and post-buckled analysis of curved beams using differential quadrature element method”, *International Journal of Mechanical and Materials Engineering*, vol. 14, no. 15, pp. 1-10.

[17] Neukirch S. Yavari M. Challamel N. Thomas O., “Comparison of the Von Kármán and Kirchhoff models for the post-buckling and vibrations of elastic beams”, *Journal of Theoretical, Computational and Applied Mechanics, INRIA*, vol. 10, pp.1-18, 2021.

[18] Wang C.Y. Wang C.M., “Vibration of a beam with an internal hinge”, *International Journal of Structural Stability and Dynamics*, vol. 1, no. 1, pp. 163-167, 2001.

[19] Lee Y.Y. Wang C.M. Kitipornchai S., “Vibration of Timoshenko beams with internal hinge”, *Journal of Engineering Mechanics*, vol. 129, no. 3, 293-301, 2003.

[20] Cheng, G.F. Lee Y.Y. Mei C., “Nonlinear random response of internally hinged beams”, *Finite Elements in Analysis and Design*, vol. 39, no. (5-6), pp. 487-504, 2003.

[21] Xiang Y. Wang C.M. Wang C.Y., “Buckling of rectangular plates with internal hinge”, *International Journal of Structural Stability and Dynamics*, vol. 1, no. 2, pp. 169-179, 2001.

[22] Xiang Y. Wang C.M. Wang C.Y. Su G.H., “Ritz buckling analysis of rectangular plates with internal hinge”, *Journal of Engineering Mechanics*, vol. 129, no. 6, 683-688, 2003.

[23] Gupta P.R. Reddy J.N., “Buckling and vibration of orthotropic plates with an internal line hinge”, *International Journal of Structural Stability and Dynamics*, vol. 2, no. 4, pp. 457-486, 2002.

[24] Wang C.M. Wang C.Y. Reddy J.N., “Exact solutions for buckling of structural members”, *CRC Series in Computational Mechanics and Applied Analysis*, 2005.

[25] Wang C.Y., “Optimum location of an internal hinge of a uniform column on an elastic foundation”, *Journal of Applied Mechanics*, vol. 75, pp. 034501(1-4), 2008.

[26] Wang C.Y., “Buckling of a weakened infinite beam on an elastic foundation”, *Journal of Engineering Mechanics*, vol. 136, No. 4, 534-537, 2010.

[27] Barari A. Kaliji H.D. Ghadimi M. Domairry G., “Non-linear vibration of Euler-Bernoulli beams”, *Latin American Journal of Solids and Structures*, vol. 8, pp. 139-148, 2011

Appendix

For *C-C* or *C-H* beam

$$W(x) = d\left[-\left(\frac{\beta}{\alpha}\right)\sin(\alpha x) - \frac{\alpha\sin(\beta) - \beta\sin(\alpha)}{\alpha(\cos(\alpha) - \cos(\beta))}\cos(\alpha x) + \sin(\beta x) + \frac{\alpha\sin(\beta) - \beta\sin(\alpha)}{\alpha(\cos(\alpha) - \cos(\beta))}\cos(\beta x)\right] \tag{A1.1}$$

$$W'(x) = d\left[-\alpha\beta\cos(\alpha x) + \alpha\beta\cos(\beta x) - \frac{\alpha\beta\sin(\alpha)\sin(\alpha x)}{\cos(\alpha) - \cos(\beta)} + \frac{\alpha^2\sin(\beta)\sin(\alpha x)}{\cos(\alpha) - \cos(\beta)} + \frac{\beta^2\sin(\alpha)\sin(\beta x)}{\cos(\alpha) - \cos(\beta)} - \frac{\alpha\beta\sin(\beta)\sin(\beta x)}{\cos(\alpha) - \cos(\beta)}\right] \tag{A1.2}$$

$$Q = \frac{1}{2}\int_0^1 (W')^2 dx \tag{A1.3}$$

$$Q = \frac{d^2}{8(\alpha^2 - \beta^2)(\cos(\alpha) - \cos(\beta))^2}(\alpha^6 + 5\alpha^4\beta^2 - 5\alpha^2\beta^4 - \beta^6 - \alpha^4\beta^2\sin(\alpha)^2 + 2\alpha^2\beta^4\sin(\alpha)^2 - \beta^6\sin(\alpha)^2 - \alpha^2\cos(\beta)^2((\alpha^2 - \beta^2)^2 - 5\alpha\beta^2\sin(2\alpha)) - 4\alpha^5\beta\sin(\alpha)\sin(\beta) + 4\alpha\beta^5\sin(\alpha)\sin(\beta) + \alpha^6\sin(\beta)^2 - 2\alpha^4\beta^2\sin(\beta)^2 + \alpha^2\beta^4\sin(\beta)^2 - 5\alpha^3\beta^2\sin(2\alpha)\sin(\beta)^2 + \beta\cos(\beta)(-4\alpha\beta(\alpha^2 + 3\beta^2)\sin(\alpha) + (-7\alpha^4 + 6\alpha^2\beta^2 + \beta^4)\sin(\beta) + (5\alpha^4 + \beta^4)\sin(\alpha)^2\sin(\beta)) + \alpha\cos(\alpha)(-8\alpha\beta^2(\alpha^2 - \beta^2)\cos(\beta) + (\alpha^4 + 5\beta^4)\cos(\beta)^2\sin(\alpha) + 4\alpha\beta(3\alpha^2 + \beta^2)\sin(\beta) - \sin(\alpha)(\alpha^4 + 6\alpha^2\beta^2 - 7\beta^4 + (\alpha^4 + 5\beta^4)\sin(\beta)^2)) + 5\alpha^2\beta^3\sin(\alpha)^2\sin(2\beta) + \beta\cos(\alpha)^2(-(5\alpha^4 + \beta^4)\cos(\beta)\sin(\beta) + \beta((\alpha^2 - \beta^2)^2 - 5\alpha^2\beta\sin(2\beta)))) \tag{A1.4}$$

Let $\left(\alpha^6 + 5\alpha^4\beta^2 - 5\alpha^2\beta^4 - \beta^6 - \alpha^4\beta^2\sin(\alpha)^2 + 2\alpha^2\beta^4\sin(\alpha)^2 - \beta^6\sin(\alpha)^2 - \alpha^2\cos(\beta)^2((\alpha^2 - \beta^2)^2 - 5\alpha\beta^2\sin(2\alpha)) - 4\alpha^5\beta\sin(\alpha)\sin(\beta) + 4\alpha\beta^5\sin(\alpha)\sin(\beta) + \alpha^6\sin(\beta)^2 - 2\alpha^4\beta^2\sin(\beta)^2 + \alpha^2\beta^4\sin(\beta)^2 - 5\alpha^3\beta^2\sin(2\alpha)\sin(\beta)^2 + \beta\cos(\beta)(-4\alpha\beta(\alpha^2 + 3\beta^2)\sin(\alpha) + (-7\alpha^4 + 6\alpha^2\beta^2 + \beta^4)\sin(\beta) + (5\alpha^4 + \beta^4)\sin(\alpha)^2\sin(\beta)) + \alpha\cos(\alpha)\left(-8\alpha\beta^2(\alpha^2 - \beta^2)\cos(\beta) + (\alpha^4 + 5\beta^4)\cos(\beta)^2\sin(\alpha) + 4\alpha\beta(3\alpha^2 + \beta^2)\sin(\beta) - \sin(\alpha)(\alpha^4 + 6\alpha^2\beta^2 - 7\beta^4 + (\alpha^4 + 5\beta^4)\sin(\beta)^2)\right) + 5\alpha^2\beta^3\sin(\alpha)^2\sin(2\beta) + \beta\cos(\alpha)^2\left(-(5\alpha^4 + \beta^4)\cos(\beta)\sin(\beta) + \beta((\alpha^2 - \beta^2)^2 - 5\alpha^2\beta\sin(2\beta))\right)\right) = M$ (A1.5)

$$\lambda = F - Q$$

$$d = \pm\frac{\sqrt{8(\alpha^2 - \beta^2)(\cos(\alpha) - \cos(\beta))^2}}{\sqrt{M}}\sqrt{F - \lambda}$$