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Analysis of Forced Response of a Damped Oscillator with Inertia and Static Nonlinearity using a Modified Lindstedt-Poincaré Method

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ABSTRACT

The study presents an analysis of periodic solutions of a highly nonlinear oscillator that includes both the inertia and static nonlinearity. In addition, the oscillator is subjected to an external excitation, thereby increasing the complexity of the system's behaviour. Recently, Alam et al. have presented a generalized modified Lindstedt-Poincaré method that covers a wide variety of nonlinear oscillators, including nonlinear oscillators with inertia and static nonlinearity. However, their method is useless if an external force is applied to the system. In this paper, an alternative modified Lindstedt-Poincaré method is proposed to investigate a damped, forced oscillator characterized by both inertial and static nonlinearities. Moreover, the resonance behaviour for various system parameters is investigated. The method is valid for both weak and strong nonlinearities. Finally, the accuracy of the analytical results is verified by comparing them with harmonic balance method (HBM) results and numerical results obtained using the Runge-Kutta fourth-order (RK4) method.

1. Introduction

Most of the perturbation methods [1-3] (including the Lindstedt-Poincaré method) were originally developed for solving weakly nonlinear differential equations

$$\ddot{x} + \omega_0^2 x + \varepsilon f(x, \dot{x}) = 0,\ \varepsilon < \omega_0 \tag{1.1}$$

where over-dots denote differentiation with respect to t, $\omega_0 > 0$ is the unperturbed frequency, ε is a small parameter, $f(x, \dot{x})$ is a nonlinear function. Then, many authors extended or modified the classical versions of various perturbation methods to handle stronger nonlinear problems. Jones [4] presented an approximate solution by introducing a parameter, $0 < \alpha(\omega_0, \varepsilon) < 1$. Cheung et al. [5] utilized this parameter to present the most useful modified version of the Lindstedt-Poincaré method. Alam et al. [6] further generalized the method, extending its applicability to a wide class of nonlinear oscillators, including those with both inertial and static nonlinearities. However, the generalized version [6] is useless when an external force is applied to the system. Recently, Alam et al. [7] investigated a nonlinear damped forced oscillator using a new analytical method similar to the modified differential transform method (MDT). The harmonic balance method [8,9] is another analytical technique for solving nonlinear oscillatory problems. However, the mathematical calculation is very difficult because a set of nonlinear algebraic equations appears when the harmonic balance method is applied. At present, many researchers [10-13] have insightfully analysed damped forced vibrations from various

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perspectives. In this work, an alternative modified Lindstedt-Poincaré method is proposed to investigate a damped, forced oscillator with both inertial and static nonlinearities. The method is valid for both weak and strong nonlinearities and exhibits excellent agreement with numerical solutions for various system parameters.

2. The Modified Lindstedt-Poincaré method [5]

For the oscillator (1.1), Cheung et al. [5] used the expansions

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{2.1}$$

and

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{2.2}$$

In the past, Veronis [14], Burton [15], and Burton and Rahman [16] also used these expansions. According to Cheung et al. [5], Eq. (2.2) can be written as

$$\omega^2 = (\omega_0^2 + \varepsilon\omega_1)\left(1 + \frac{\varepsilon^2\omega_2}{(\omega_0^2+\varepsilon\omega_1)} + \cdots\right). \tag{2.3}$$

By defining a new parameter $\alpha = \frac{\varepsilon\omega_1}{(\omega_0^2+\varepsilon\omega_1)}$, Eq. (2.3) becomes

$$\omega^2 = \frac{\omega_0^2}{1-\alpha}(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots). \tag{2.4}$$

Cheung et al. [5] also reconsidered Eq. (2.1) as follows

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots \tag{2.5}$$

Substituting Eqs. (2.4) and (2.5) into Eq. (1.1) and equating the coefficients of α, yields a set of linear equations, which can then be solved using perturbation steps.

Now, let us consider the nonlinear Duffing oscillator with linear damping and external excitation, given by

$$\ddot{x} + \omega_0^2 x + \varepsilon\mu\dot{x} + \varepsilon x^3 = \varepsilon E\cos(\nu t),\ x(0) = a, \tag{2.6}$$

where E is the forcing amplitude, and ν is the forcing frequency, known as the driving frequency.

Using the transformation $\tau = \nu t$, Eq. (2.5) can be written as

$$\nu^2 x'' + \omega_0^2 x + \varepsilon\mu\nu x' + \varepsilon x^3 = \varepsilon E\cos\tau, \tag{2.7}$$

where the forcing frequency can be expanded as

$$\nu^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{2.8}$$

Also, using the same parameter $\alpha = \frac{\varepsilon\omega_1}{(\omega_0^2+\varepsilon\omega_1)}$ or $= \frac{\omega_0^2\alpha}{\omega_1(1-\alpha)}$, the forcing frequency ν^2 can be expressed in terms of α as

$$\nu^2 = \frac{\omega_0^2}{1-\alpha}(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots). \tag{2.9}$$

Substituting Eqs. (2.5) and (2.9) into Eq. (2.7), and then equating the coefficient of various powers of α, the following linear equations are obtained

$$x_0'' + x_0 = 0, \tag{2.10}$$

$$x_1'' + x_1 = x_0 - \frac{1}{\omega_1}x_0^3 - \frac{\mu\omega_0}{\omega_1}x_0' + \frac{E}{\omega_1}\cos\tau, \tag{2.11}$$

$$x_2'' + x_2 = x_1 - \frac{1}{\omega_1}3x_0^2 x_1 - \delta_2 x_0'' - \frac{\mu\omega_0}{\omega_1}x_1' - \frac{\mu\omega_0}{2\omega_1}x_0', \tag{2.12}$$

where the initial conditions given in Eq. (2.6) are changed to

$$x_0(0) = a_0, x_n(0) = 0, n = 1, 2, \cdots. \tag{2.13}$$

The solution of Eq. (2.10) is considered as

$$x_0 = a_0\cos\tau + b_0\sin\tau. \tag{2.14}$$

Solving Eqs. (2.11) and (2.12), the following results are obtained

$$\omega_1 = \frac{3}{4}({a_0}^2 + {b_0}^2) - \left(\frac{E}{a_0} - \frac{b_0\mu}{a_0}\omega_0\right), \tag{2.15}$$

$$b_0 = \frac{-E + \sqrt{E^2 - 4\mu^2{\omega_0}^2{a_0}^2}}{2\mu\omega_0}, \tag{2.16}$$

$$\begin{aligned}\delta_2 = {} & \frac{1}{4\omega_1 a_0}a_1(9{a_0}^2 + 3{b_0}^2 - 4\omega_1) \\ & + \frac{1}{2\omega_1 a_0}b_1(2\mu\omega_0 + 3a_0b_0) + \frac{1}{2\omega_1 a_0}\mu\omega_0 b_0 \\ & + \frac{3}{128{\omega_1}^2}({a_0}^2 - 3{b_0}^2)({a_0}^2 - {b_0}^2) \\ & + \frac{3}{64{\omega_1}^2}{b_0}^2(3{a_0}^2 - {b_0}^2),\end{aligned} \tag{2.17}$$

$$\begin{aligned}x_1 = {} & a_1\cos\tau + b_1\sin\tau \\ & + \frac{a_0}{32\omega_1}\left({a_0}^2 - 3{b_0}^2\right)\cos 3\tau \\ & + \frac{b_0}{32\omega_1}\left(3{a_0}^2 - {b_0}^2\right)\sin 3\tau.\end{aligned} \tag{2.18}$$

Here, the values of a_1 and b_1 are to be determined using initial conditions (2.13) as

$$a_1 = \frac{-a_0}{32\omega_1}({a_0}^2 - 3{b_0}^2), \tag{2.19}$$

$$b_1 = \frac{1}{\Lambda}(a_0 D_{23} - b_0 D_{13}), \tag{2.20}$$

where

$$\Lambda = \frac{1}{4\omega_1}(4\omega_1 a_0 + 4b_0\mu - 3{a_0}^3 - 3a_0{b_0}^2), \tag{2.21}$$

$$\begin{aligned}D_{13} = {} & \frac{1}{4\omega_1}a_1(9{a_0}^2 + 3{b_0}^2 - 4\omega_1) + \frac{b_0\mu\omega_0}{2\omega_1} \\ & + \frac{3}{128{\omega_1}^2}a_0({a_0}^2 - 3{b_0}^2)({a_0}^2 - {b_0}^2) \\ & + \frac{3}{64{\omega_1}^2}{a_0}^2{b_0}^2(3{a_0}^2 - {b_0}^2),\end{aligned} \tag{2.22}$$

$$\begin{aligned}D_{23} = {} & \frac{1}{4\omega_1}a_1(6a_0b_0 - 4\mu\omega_0) - \frac{a_0\mu\omega_0}{2\omega_1} \\ & - \frac{3}{128{\omega_1}^2}b_0(3{a_0}^2 - {b_0}^2)({b_0}^2 - {a_0}^2) \\ & + \frac{3}{64{\omega_1}^2}{a_0}^2 b_0({a_0}^2 - 3{b_0}^2).\end{aligned} \tag{2.23}$$

The first approximate amplitude can be obtained by

$$R = \sqrt{{A_1}^2 + {B_1}^2}, \tag{2.24}$$

where

$$A_1 = a_0 + \alpha a_1, \tag{2.25}$$

and

$$B_1 = b_0 + \alpha b_1. \tag{2.26}$$

The preceding mathematical analysis demonstrates that the first-order approximation becomes notably cumbersome, and this difficulty is expected to increase excessively in the subsequent higher-order approximations. Predictably, the modified Lindstedt-Poincaré method by Cheung et al. [5] will become more difficult to apply to the damped forced oscillator where the inertia force is present in the nonlinear function, due to its complicated nature.

3. The proposed modified Lindstedt-Poincaré method

Consider a damped forced vibration system given by

$$\ddot{x} + \omega_0^2 x + \varepsilon\mu\dot{x} + \varepsilon f(x) = \varepsilon E\,\Phi(\nu t), \tag{3.1}$$

with the initial condition

$$x(0) = a_0, \tag{3.2}$$

where E and ν are the amplitude and frequency of the periodic function $\Phi(\nu t)$, respectively.

Consider a new variable $\tau = \nu t - \varphi$, where φ is the phase parameter. Eq. (3.1) can be written as

$$\nu^2 x'' + \omega_0^2 x + \varepsilon\mu\nu x' + \varepsilon f(x) = \varepsilon E\,\Phi(\tau + \varphi). \tag{3.3}$$

The approximate solution x, fundamental frequency ν, and the phase φ should be expanded in a power series of ε as

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots, \tag{3.4}$$

$$\nu^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{3.5}$$

and

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots. \tag{3.6}$$

Letting $\mu_1 = \mu\nu$, then substituting Eqs. (3.4) to (3.6) into Eq. (3.3) and equating the coefficients of like powers of ε, the following linear equations are obtained

$$x_0'' + x_0 = 0, \tag{3.7}$$

$$\omega_0^2(x_1'' + x_1) = x_0''\omega_1 - f(x_0) - \mu_1 x_0' + Eg(\tau + \varphi_0), \tag{3.8}$$

$$\omega_0^2(x_2'' + x_2) = -\mu_1 x_1' - \omega_1 x_1'' - \omega_2 x_0'' - x_1\frac{\partial f(x_0)}{\partial x_0} + E\varphi_1\frac{\partial}{\partial\tau}g(\tau + \varphi_0), \tag{3.9}$$

where the initial conditions are

$$x_0(0) = a, x_0'(0) = 0, x_1(0) = 0, x_1'(0) = 0,$$
$$x_2(0) = 0, x_2'(0) = 0, \cdots. \tag{3.10}$$

The above linear equations are solved using the standard Lindstedt-Poincaré method, which yields the successive approximations $x_0, x_1, x_2, \cdots$. By eliminating the secular terms, the frequency coefficients $\omega_1, \omega_2, \omega_3 \cdots$, and the phase coefficients $\varphi_0, \varphi_1, \varphi_{2\ldots}$ are obtained.

4. Application to the damped forced oscillator with inertia and static nonlinearity

Let us consider an excited damped oscillator having inertia and static nonlinearity

$$\ddot{x} + \omega_0^2 x + \varepsilon\mu\dot{x} + \varepsilon\beta x^2\ddot{x} + \varepsilon\gamma\dot{x}^2 x + \varepsilon\lambda x^3 = \varepsilon E\cos(\nu t), \tag{4.1}$$

with the initial condition

$$x(0) = a_0, \tag{4.2}$$

where E and ν are the forcing amplitude and frequency, respectively.

Letting $\tau = \nu t - \varphi$, Eq. (4.1) becomes

$$\nu^2 x'' + \omega_0^2 x + \varepsilon\mu_1 x' + \varepsilon\nu^2\beta x^2 x'' + \varepsilon\nu^2\gamma x'^2 x + \varepsilon\lambda x^3 = \varepsilon E\cos(\tau + \varphi), \tag{4.3}$$

where $\mu_1 = \mu\nu$ and the initial conditions are considered as

$$x(0) = a,\ x'(0) = 0. \tag{4.4}$$

Substituting Eqs. (3.4) to (3.6) into Eq. (4.3), and then equating the coefficients of like powers of ε, the following linear equations are obtained

$$x_0'' + x_0 = 0, \tag{4.5}$$

$$\omega_0^2(x_1'' + x_1) = -x_0''\omega_1 - \lambda x_0^3 - \mu_1 x_0' - \beta\omega_0^2 x_0^2 x_0'' - \gamma\omega_0^2 x_0 {x_0'}^2 + E\cos(\tau + \varphi_0), \tag{4.6}$$

$$\omega_0^2(x_2'' + x_2) = -3\lambda x_0^2 x_1 - \mu_1 x_1' - 2x_0 x_1 x_0''\beta\omega_0^2 + {x_0}^2 x_1''\beta\omega_0^2 + x_1 {x_0'}^2\gamma\omega_0^2 + 2x_0 x_0' x_1'\gamma\omega_0^2 + {x_0}^2 x_0''\beta\omega_1 + x_0 {x_0'}^2\gamma\omega_1 - \omega_2 x_0'' - \omega_1 x_1'' - E\varphi_1\sin(\tau + \varphi_0), \tag{4.7}$$

where

$$x_0(0) = a, x_0'(0) = 0, x_1(0) = 0, x_1'(0) = 0,$$
$$x_2(0) = 0, x_2'(0) = 0, \cdots. \tag{4.8}$$

The solution of Eq. (4.5) is

$$x_0 = a\cos\tau, \tag{4.9}$$

Substituting x_0 into Eq. (4.6) yields

$$\omega_0^2(x_1'' + x_1) = \left(a\omega_1 - \frac{3a^3\lambda}{4} + \frac{3}{4}a^3\beta\omega_0^2 - \frac{1}{4}a^3\gamma\omega_0^2 + E\cos\varphi_0\right)\cos\tau + (a\mu_1 - E\sin\varphi_0)\sin\tau - \left(\frac{a^3\lambda}{4} - \frac{1}{4}a^3\beta\omega_0^2 - \frac{1}{4}a^3\gamma\omega_0^2\right)\cos 3\tau. \tag{4.10}$$

Eliminating the secular terms, the following results are obtained

$$\varphi_0 = \sin^{-1}\left(\frac{a\mu_1}{E}\right), \tag{4.11}$$

$$\omega_1 = \frac{3a^2\lambda}{4} - \frac{3}{4}a^2\beta\omega_0^2 + \frac{1}{4}a^2\gamma\omega_0^2 - \frac{E_0}{a}, \tag{4.12}$$

where

$$E_0 = \sqrt{E^2 - a^2{\mu_1}^2}. \tag{4.13}$$

Now, Eq. (4.10) takes the form

$$\omega_0^2(x_1'' + x_1) = -\left(\frac{a^3\lambda}{4} - \frac{1}{4}a^3\beta\omega_0^2 - \frac{1}{4}a^3\gamma\omega_0^2\right)\cos 3\tau. \tag{4.14}$$

Solving Eq. (4.14) for x_1 with the initial conditions given in Eq. (4.8), yields

$$x_1 = \frac{1}{8\omega_0^2}\left(\frac{a^3\lambda}{4} - \frac{1}{4}a^3\beta\omega_0^2 - \frac{1}{4}a^3\gamma\omega_0^2\right)\cos 3\tau \tag{4.15}$$

Again, substituting x_0 and x_1 into Eq. (4.7) and solving, yields the following results

$$\varphi_1 = \frac{a^3\mu_1(-\lambda + (\beta+\gamma)\omega_0^2)}{32\mathrm{E}1\omega_0^2}, \tag{4.16}$$

$$\omega_2 = \frac{a\left(4a^2{\mu_1}^2(-\lambda + (\beta+\gamma)\omega_0^2) - 4\mathrm{E}^2(\lambda + (-25\beta + 7\gamma)\omega_0^2)\right)}{128\mathrm{E}\omega_0^2} + \frac{a^4\mathrm{E}\left(-3\lambda^2 + 2(-37\beta + 15\gamma)\lambda\omega_0^2 + (77\beta^2 - 46\beta\gamma + 5\gamma^2)\omega_0^4\right)}{128\mathrm{E}\omega_0^2}, \tag{4.17}$$

$$x_2 = \frac{1}{256\omega_0^4}\left(\left(-\frac{11}{2}a^5\beta\lambda + \frac{1}{2}a^5\gamma\lambda + \frac{3a^5\lambda^2}{4\omega_0^2} + \frac{19}{4}a^5\beta^2\omega_0^2 + \frac{7}{2}a^5\beta\gamma\omega_0^2 - \frac{5}{4}a^5\gamma^2\omega_0^2\right) + \left(a^3\beta\omega_1 + a^3\gamma\omega_1 - \frac{9a^3\lambda\omega_1}{\omega_0^2}\right)\right)(\cos 3\tau - \cos\tau)$$
$$+ \frac{1}{3072\omega_0^4}\left(\left(-14a^5\beta\lambda - 10a^5\gamma\lambda + \frac{3a^5\lambda^2}{\omega_0^2} + 11a^5\beta^2\omega_0^2 + 18a^5\beta\gamma\omega_0^2\right) + 7a^5\gamma^2\omega_0^2\right)(\cos 5\tau - \cos\tau)$$

$$+\frac{1}{256\omega_0^4}(3a^3\beta\mu_1 + 3a^3\gamma\mu_1 - \frac{3a^3\lambda\mu_1}{\omega_0^2})(\sin 3\tau - 3\sin\tau). \quad (4.18)$$

In a similar procedure, the higher-order approximations can be determined.

5. Results and discussion

In this paper, a modified Lindstedt-Poincaré method is applied to investigate nonlinear oscillators, applicable to both autonomous and non-autonomous systems satisfying $f(-x,-\dot{x}) = -f(x,\dot{x})$. Here, we consider a complicated nonlinear damped oscillator with inertia and static nonlinearity subjected to an external excitation. When $\gamma = 0$, the oscillator becomes simple, known as the damped forced vibration of the Duffing oscillator. Here, $\gamma = 1$ is considered. The existing modified Lindstedt-Poincaré methods are less effective in solving this oscillator. Moreover, even the first-order approximation, Cheung et al. [5] modified version of the Lindstedt-Poincaré method, yields a large number of laborious calculations. On the contrary, the proposed modified version of the Lindstedt-Poincaré method is very simple and effective. The method provides steady-state solutions for nonlinear damped forced oscillators with both inertial and static nonlinearities, and remains effective under strong nonlinearities and various damping conditions.

The frequency-amplitude relationship of Eq. (4.1) for various system parameters is shown in Table 5.1 and Fig. 5.1-5.3. The table shows that the present results are in good agreement with numerical results (RK4). The figures also show that the present results are in excellent agreement with the harmonic balance method (HBM) results and the corresponding numerical results. Latter, the time-displacement response curves for various parameters of Eq. (4.1) are plotted in Fig. 5.4-5.7. It is observed that the present solutions are also in excellent agreement with the HBM solution and the corresponding numerical solution. In addition, a comparison of the phase-plane between the present method and the numerical method is shown in Fig. 5.8-5.11. From the figures, it is observed that the present method is highly efficient for solving nonlinear damped forced vibrations when both inertial and static forces are present in the nonlinear function.

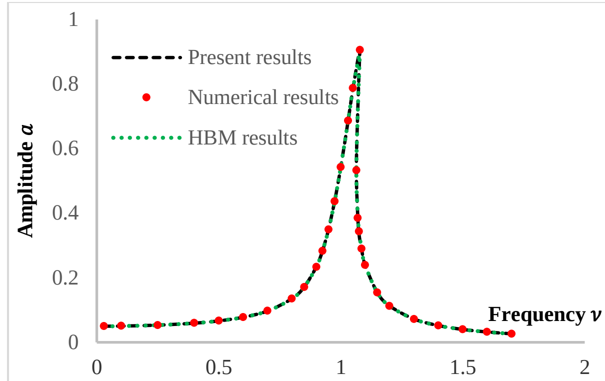

Figure 5.1: Comparison of frequency-amplitude responses of Eq. (4.1) for $\mu = 0.05, \lambda = 1, \omega_0 = 1, \gamma = 1, \beta = 1, E = 0.05$.

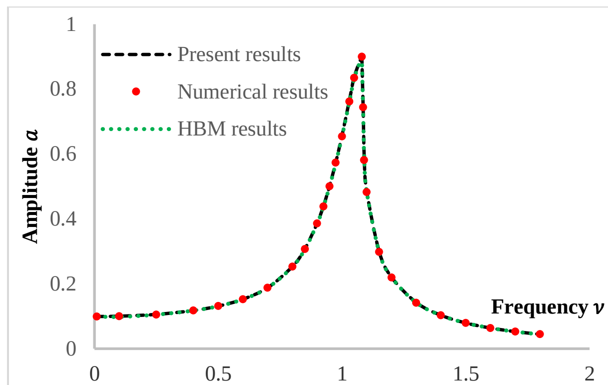

Figure 5.2: Comparison of frequency-amplitude responses of Eq. (4.1) for $\mu = 0.1, \lambda = 1, \omega_0 = 1, \gamma = 1, \beta = 1, E = 0.1$.

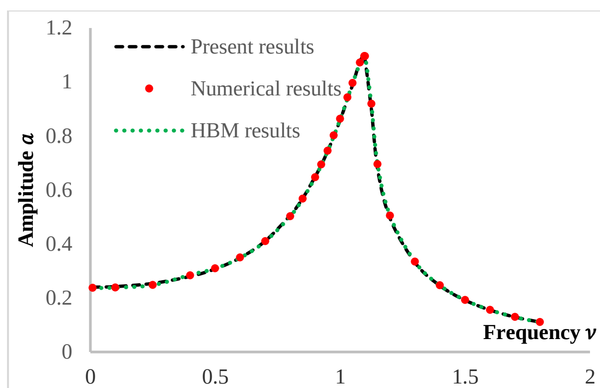

Figure 5.3: Comparison of frequency-amplitude responses of Eq. (4.1) for $\mu = 0.2, \lambda = 1, \omega_0 = 1, \gamma = 1, \beta = 1, E = 0.25$.

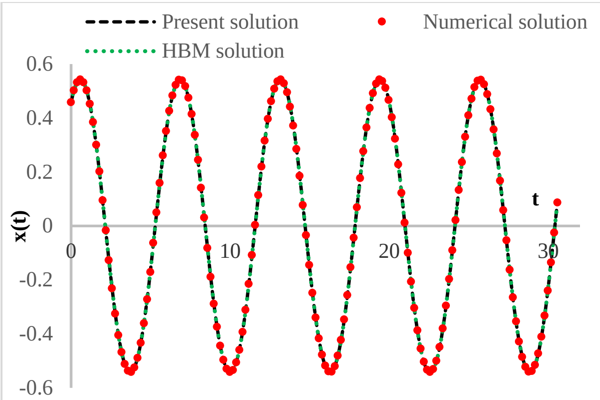

Figure 5.4: Comparison of time-displacement responses of Eq. (4.1) for $\mu = 0.05, \gamma = 1, \lambda = 1, \nu = 1,\ \omega_0 = 1, \beta = 1, E = 0.05$, and $a = 0.54228$.

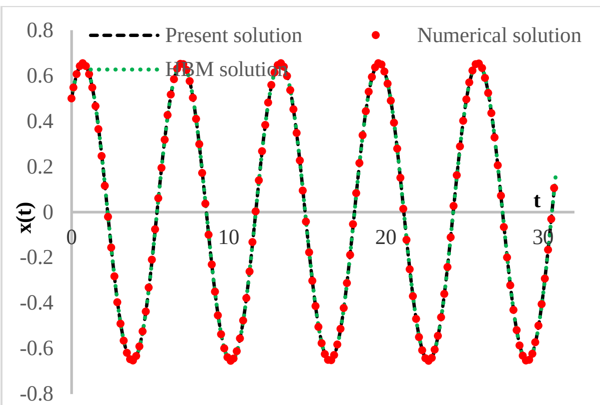

Figure 5.5: Comparison of time-displacement responses of Eq. (4.1) for $\mu = 0.1, \gamma = 1, \lambda = 1, \nu = 1,\ \omega_0 = 1, \beta = 1, E = 0.1$, and $a = 0.65387$.

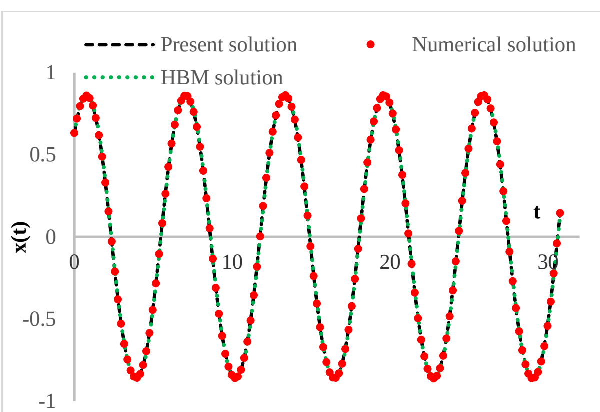

Figure 5.6: Comparison of time-displacement responses of Eq. (4.1) for $\mu = 0.2, \gamma = 1, \lambda = 1, \nu = 1,\ \omega_0 = 1, \beta = 1, E = 0.25$, and $a = 0.85931$.

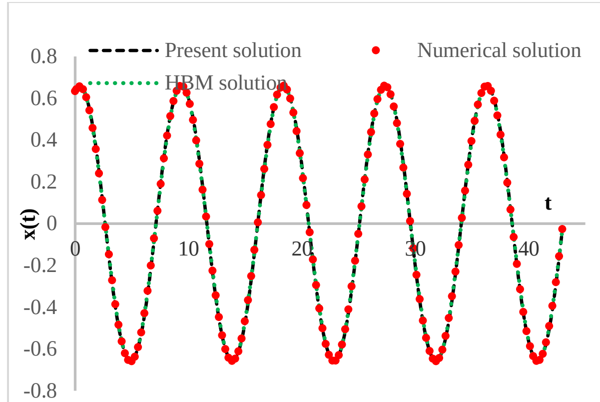

Figure 5.7: Comparison of time-displacement responses of Eq. (4.1) for $\mu = 0.3, \gamma = 1, \lambda = 1, \nu = 1,\ \omega_0 = 1, \beta = 1, E = 0.5$, and $a = 0.65595$.

Table 5.1: Comparison of present amplitudes of Eq. (4.1) with the amplitudes obtained using the RK4 for various system parameters.

Parameters	ν	RK4 a	Present a (Err%)
$E = 0.05$ $\mu = 0.05$ $\beta = 1$ $\gamma = 1$ $\lambda = 1$ $\omega_0 = 1$	0.1	0.0503738	0.0504114 (0.07)
	0.25	0.0531389	0.0532163 (0.15)
	0.4	0.0594199	0.0593432 (0.13)
	0.6	0.0776382	0.0776232 (0.02)
	0.8	0.135091	0.135085 (0.00)
	1.0	0.542419	0.54228 (0.03)
$E = 0.1$ $\mu = 0.1$ $\beta = 1$ $\gamma = 1$ $\lambda = 1$ $\omega_0 = 1$	0.1	0.0999845	0.100273 (0.29)
	0.25	0.105165	0.105749 (0.55)
	0.4	0.11821	0.117641 (0.48)
	0.6	0.152548	0.152438 (0.07)
	0.8	0.252517	0.252481 (0.01)
	1.0	0.654605	0.653876 (0.11)
$E = 0.25$ $\mu = 0.2$ $\beta = 1$ $\gamma = 1$ $\lambda = 1$ $\omega_0 = 1$	0.1	0.238639	0.242231 (1.51)
	0.25	0.247352	0.254176 (2.76)
	0.4	0.285918	0.279435 (2.27)
	0.6	0.348915	0.347775 (0.32)
	0.8	0.502408	0.502071 (0.07)
	1.0	0.862603	0.854772 (0.91)

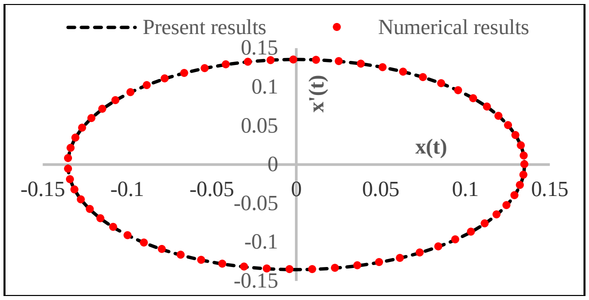

Figure 5.8: Comparison of phase plane of Eq. (4.1) for $\mu = 0.05, \gamma = 1, \lambda = 1, \nu = 0.8,\ \omega_0 = 1, \beta = 1, E = 0.05$.

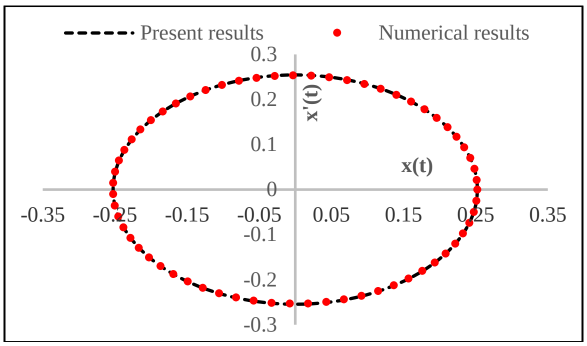

Figure 5.9: Comparison of phase plane of Eq. (4.1) for $\mu = 0.1, \gamma = 1, \lambda = 1, \nu = 0.8,\ \omega_0 = 1, \beta = 1, E = 0.1$.

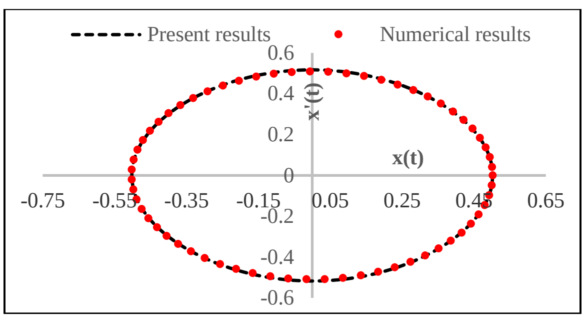

Figure 5.10: Comparison of phase plane of Eq. (4.1) for $\mu = 0.2, \gamma = 1, \lambda = 1, \nu = 0.8,\ \omega_0 = 1, \beta = 1, E = 0.25$.

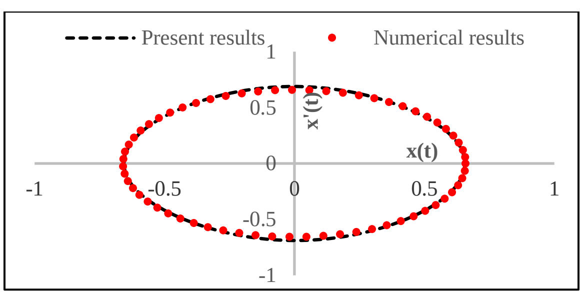

Figure 5.11: Comparison of phase plane of Eq. (4.1) for $\mu = 0.3, \gamma = 1, \lambda = 1, \nu = 0.7,\ \omega_0 = 1, \beta = 1, E = 0.5$.

Conclusion

A modified Lindstedt-Poincaré method is presented to find a steady-state solution for a damped forced oscillator with inertia and static nonlinearity. The present method is simpler and more effective than the existing Lindstedt-Poincaré method for investigating nonlinear oscillators with strong nonlinearities and various damping conditions. The approximate results determined by the present method show surprisingly good agreement with those obtained by the numerical method.

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